

# PERMUTATION AND COMBINATION

## 1. FUNDAMENTAL PRINCIPLES OF COUNTING

### 1.1 Fundamental Principle of Multiplication

If an event can occur in  $m$  different ways following which another event can occur in  $n$  different ways following which another event can occur in  $p$  different ways. Then the total number of ways of simultaneous happening of all these events in a definite order is  $m \times n \times p$ .

### 1.2 Fundamental Principle of Addition

If there are two jobs such that they can be performed independently in  $m$  and  $n$  ways respectively, then either of the two jobs can be performed in  $(m + n)$  ways.

## 2. SOME BASIC ARRANGEMENTS AND SELECTIONS

### 2.1 Combinations

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed, is called a combination.

### 2.2 Permutations

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.



- Let  $r$  and  $n$  be positive integers such that  $1 \leq r \leq n$ . Then, the number of all permutations of  $n$  distinct items or objects taken  $r$  at a time, is

$${}^n P_r = {}^n C_r \times r!$$

Proof : Total ways =  $n(n-1)(n-2) \dots (n-r+1)$

$$= \frac{n(n-1)(n-2) \dots (n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

$$= {}^n P_r.$$

So, the total no. of arrangements (permutations) of  $n$ -distinct items, taking  $r$  at a time is  ${}^n P_r$  or  $P(n, r)$ .

- The number of all permutations (arrangements) of  $n$  distinct objects taken all at a time is  $n!$ .
- The number of ways of selecting  $r$  items or objects from a group of  $n$  distinct items or objects, is

$$\frac{n!}{(n-r)!r!} = {}^n C_r.$$

### 3. GEOMETRIC APPLICATIONS OF ${}^n C_r$

- (i) Out of  $n$  non-concurrent and non-parallel straight lines, points of intersection are  ${}^n C_2$ .
- (ii) Out of ' $n$ ' points the number of straight lines are (when no three are collinear)  ${}^n C_2$ .
- (iii) If out of  $n$  points  $m$  are collinear, then No. of straight lines =  ${}^n C_2 - {}^m C_2 + 1$
- (iv) In a polygon total number of diagonals out of  $n$  points (no three are collinear) =  ${}^n C_2 - n = \frac{n(n-3)}{2}$ .
- (v) Number of triangles formed from  $n$  points is  ${}^n C_3$ . (when no three points are collinear)
- (vi) Number of triangles out of  $n$  points in which  $m$  are collinear, is  ${}^n C_3 - {}^m C_3$ .
- (vii) Number of triangles that can be formed out of  $n$  points (when none of the side is common to the sides of polygon), is  ${}^n C_3 - {}^n C_1 - {}^n C_1 \cdot {}^{n-4} C_1$
- (viii) Number of parallelograms in two systems of parallel lines (when 1<sup>st</sup> set contains  $m$  parallel lines and 2<sup>nd</sup> set contains  $n$  parallel lines), is =  ${}^n C_2 \times {}^m C_2$
- (ix) Number of squares in two system of perpendicular parallel lines (when 1<sup>st</sup> set contains  $m$  equally spaced parallel lines and 2<sup>nd</sup> set contains  $n$  same spaced parallel lines)

$$= \sum_{r=1}^{m-1} (m-r)(n-r); (m < n)$$

### 4. PERMUTATIONS UNDER CERTAIN CONDITIONS

The number of all permutations (arrangements) of  $n$  different objects taken  $r$  at a time :

- (i) When a particular object is to be always included in each arrangement, is  ${}^{n-1} C_{r-1} \times r!$ .
- (ii) When a particular object is never taken in each arrangement, is  ${}^{n-1} C_r \times r!$ .

### 5. DIVISION OF OBJECTS INTO GROUPS

#### 5.1 Division of items into groups of unequal sizes

1. The number of ways in which  $(m + n)$  distinct items can be divided into two unequal groups containing

$$m \text{ and } n \text{ items, is } \frac{(m+n)!}{m!n!}.$$

2. The number of ways in which  $(m+ n+ p)$  items can be divided into unequal groups containing  $m, n, p$  items, is

$${}^{m+n+p} C_m \cdot {}^{n+p} C_m = \frac{(m+n+p)!}{m!n!p!}.$$

3. The number of ways to distribute  $(m+n+p)$  items among 3 persons in the groups containing  $m, n$  and  $p$  items

$$= (\text{No. of ways to divide}) \times (\text{No. of groups})!$$

$$= \frac{(m+n+p)!}{m!n!p!} \times 3!$$

#### 5.2 Division of Objects into groups of equal size

The number of ways in which  $mn$  different objects can be divided equally into  $m$  groups, each containing  $n$  objects and the order of the groups is not important, is

$$\left( \frac{(mn)!}{(n!)^m} \right) \frac{1}{m!}$$

The number of ways in which  $mn$  different items can be divided equally into  $m$  groups, each containing  $n$  objects and the order of groups is important, is

$$\left( \frac{(mn)!}{(n!)^m} \times \frac{1}{m!} \right) m! = \frac{(mn)!}{(n!)^m}$$

**6. PERMUTATIONS OF ALIKE OBJECTS**

1. The number of mutually distinguishable permutations of  $n$  things, taken all at a time, of which  $p$  are alike of one kind,  $q$  alike of second kind such that  $p + q = n$ , is

$$\frac{n!}{p!q!}$$

2. The number of permutations of  $n$  things, of which  $p$  are alike of one kind,  $q$  are alike of second kind and

remaining all are distinct, is  $\frac{n!}{p!q!}$ . Here  $p + q \neq n$

3. The number of permutations of  $n$  things, of which  $p_1$  are alike of one kind;  $p_2$  are alike of second kind;  $p_3$  are alike of third kind; ..... ;  $p_r$  are alike of  $r^{\text{th}}$  kind such that

$$p_1 + p_2 + \dots + p_r = n, \text{ is } \frac{n!}{p_1!p_2!p_3!\dots p_r!}$$

4. Suppose there are  $r$  things to be arranged, allowing repetitions. Let further  $p_1, p_2, \dots, p_r$  be the integers such that the first object occurs exactly  $p_1$  times, the second occurs exactly  $p_2$  times subject, etc. Then the total number of permutations of these  $r$  objects to the above condition, is

$$\frac{(p_1 + p_2 + \dots + p_r)!}{p_1!p_2!p_3!\dots p_r!}$$

**7. DISTRIBUTION OF ALIKE OBJECTS**

- (i) The total number of ways of dividing  $n$  identical items among  $r$  persons, each one of whom, can receive 0, 1, 2, or more items ( $\leq n$ ), is  ${}^{n+r-1}C_{r-1}$ .

OR

The total number of ways of dividing  $n$  identical objects into  $r$  groups, if blank groups are allowed, is  ${}^{n+r-1}C_{r-1}$ .

- (ii) The total number of ways of dividing  $n$  identical items among  $r$  persons, each of whom, receives at least one item is  ${}^{n-1}C_{r-1}$ .

OR

The number of ways in which  $n$  identical items can be divided into  $r$  groups such that blank groups are not allowed, is  ${}^{n-1}C_{r-1}$ .

- (iii) The number of ways in which  $n$  identical items can be divided into  $r$  groups so that no group contains less than  $k$  items and more than  $m$  ( $m < k$ ) is

The coefficient of  $x^n$  in the expansion of  $(x^m + x^{m+1} + \dots + x^k)^r$

**8. NO. OF INTEGRAL SOLUTIONS OF LINEAR EQUATIONS AND INEQUATIONS**

Consider the eqn.  $x_1 + x_2 + x_3 + x_4 + \dots + x_r = n$  ... (i)

where  $x_1, x_2, \dots, x_r$  and  $n$  are non-negative integers.

This equation may be interpreted as that  $n$  identical objects are to be divided into  $r$  groups.

1. The total no. of non-negative integral solutions of the equation  $x_1 + x_2 + \dots + x_r = n$  is  ${}^{n+r-1}C_{r-1}$ .
2. The total number of solutions of the same equation in the set  $N$  of natural numbers is  ${}^{n-1}C_{r-1}$ .

3. In order to solve inequations of the form

$$x_1 + x_2 + \dots + x_m \leq n$$

we introduce a dummy (artificial) variable  $x_{m+1}$  such that  $x_1 + x_2 + \dots + x_m + x_{m+1} = n$ , where  $x_{m+1} \geq 0$ .

The no. of solutions of this equation are same as the no. of solutions of in Eq. (i).

### 9. CIRCULAR PERMUTATIONS

1. The number of circular permutations of  $n$  distinct objects is  $(n - 1)!$ .
2. If anti-clockwise and clockwise order of arrangements are not distinct then the number of circular permutations of  $n$  distinct items is  $1/2 \{(n - 1)!\}$   
e.g., arrangements of beads in a necklace, arrangements of flowers in a garland etc.

### 10. SELECTION OF ONE OR MORE OBJECTS

1. The number of ways of selecting one or more items from a group of  $n$  distinct items is  $2^n - 1$ .

**Proof :** Out of  $n$  items, 1 item can be selected in  ${}^n C_1$  ways; 2 items can be selected in  ${}^n C_2$  ways; 3 items can be selected in  ${}^n C_3$  ways and so on.....

Hence, the required number of ways

$$\begin{aligned} &= {}^n C_1 + {}^n C_2 + {}^n C_3 + \dots + {}^n C_n \\ &= ({}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_n) - {}^n C_0 \\ &= 2^n - 1. \end{aligned}$$

2. The number of ways of selecting  $r$  items out of  $n$  identical items is 1 .
3. The total number of ways of selecting zero or more items from a group of  $n$  identical items is  $(n + 1)$ .
4. The total number of selections of some or all out of  $p + q + r$  items where  $p$  are alike of one kind,  $q$  are alike of second kind and rest are alike of third kind, is  $[(p + 1)(q + 1)(r + 1)] - 1$ .
5. The total number of ways of selecting one or more items from  $p$  identical items of one kind;  $q$  identical items of second kind;  $r$  identical items of third kind and  $n$  different items, is  $(p + 1)(q + 1)(r + 1) 2^n - 1$

### 11. THE NUMBER OF DIVISORS AND THE SUM OF THE DIVISORS OF A GIVEN NATURAL NUMBER

$$\text{Let } N = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k} \quad \dots(1)$$

where  $p_1, p_2, \dots, p_k$  are distinct prime numbers and  $n_1, n_2, \dots, n_k$  are positive integers.

1. Total number of divisors of  $N = (n_1 + 1)(n_2 + 1) \dots (n_k + 1)$ .
2. This includes 1 and  $n$  as divisors. Therefore, number of divisors other than 1 and  $n$ , is

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1) - 2.$$

3. The sum of all divisors of (1) is given by

$$= \left\{ \frac{p_1^{n_1+1} - 1}{p_1 - 1} \right\} \left\{ \frac{p_2^{n_2+1} - 1}{p_2 - 1} \right\} \left\{ \frac{p_3^{n_3+1} - 1}{p_3 - 1} \right\} \dots \left\{ \frac{p_k^{n_k+1} - 1}{p_k - 1} \right\}.$$

### 12. DEARRANGEMENTS

If  $n$  distinct objects are arranged in a row, then the no. of ways in which they can be dearranged so that none of them occupies its original place, is

$$n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!} \right\}$$

and it is denoted by  $D(n)$ .

If  $r$  ( $0 \leq r \leq n$ ) objects occupy the places assigned to them i.e., their original places and none of the remaining  $(n - r)$  objects occupies its original places, then the no. of such ways, is

$$\begin{aligned} D(n - r) &= {}^n C_r \cdot D(n - r) \\ &= {}^n C_r \cdot (n - r)! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-r} \frac{1}{(n - r)!} \right\}. \end{aligned}$$